MA 410, Fall 2011

Assignment 1

This homework is due *Friday*, September 16.

There are total 25 points in this assignment. 22 points is considered 100%. If you go over 22 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations. Bare answers will not earn you much.

This assignment covers sections 1.3–1.5 in O'Neill.

(1) (Exercise 1.3.3abcef in O'Neill) Let $V = y^2U_1 - xU_3$, and let $f = xy$, $g = z^3$. Compute the functions
  (a) [1pt] $V[f]$, $V[g]$,
  (b) [1pt] $V[fg]$, $V[f^2 + g^2]$,
  (c) [1pt] $V[V[f]]$.

(2) [2pt] (1.3.4) Prove the identity $V = \sum V[x_i]U_i$, where x_1, x_2, x_3 are the euclidean coordinate functions. (*Hint:* using $V = \sum v_iU_i$, evaluate V on x_j.)

(3) [2pt] (1.3.5) If $V[f] = W[f]$ for every function f on $\mathbb{R}^3$, prove that $V = W$. (*Hint:* consider $f = x_i$.)

(4) [1pt] (1.4.1) Compute the velocity vector of the curve $\alpha(t) = (1+\cos t, \sin t, 2\sin\frac{t}{2})$ for arbitrary t. Further, for $t = 0$, $t = \pi/2$, $t = \pi$ draw the velocity vector on the sketch of the curve (see drawing in lectures or fig. 1.8 in O'Neill).

(5) [1pt] (1.4.2) Find the unique curve such that $\alpha(0) = (1, 0, 5)$ and $\alpha'(t) = (t^2, t, e^t)$.

(6) [2pt] (1.4.4) Reparametrize curve $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$ using $h(s) = \log s$ on $J : s > 0$. Check the chain rule for this reparametrization $\beta = \alpha(h)$ explicitly, that is, compute separately both sides of equality

$$\beta'(s) = \frac{dh}{ds}(s) \cdot \alpha'(h(s))$$

and make sure they are actually equal.

— *see next page* —

(7) (1.4.6–7) Consider three curves

$$\alpha = (t, 1+t^2, t), \quad \beta = (\sin t, \cos t, t), \quad \gamma = (\sinh t, \cosh t, t).$$

Note that at $t = 0$ they pass through the same point.

(a) [1pt] For $f = x^2 - y^2 + z^2$, compute (by a straightforward calculation) $\frac{d(f(\alpha))}{dt}|_{t=0}$, $\frac{d(f(\beta))}{dt}|_{t=0}$, $\frac{d(f(\beta))}{dt}|_{t=0}$.

(b) [2pt] Explain how one could tell that these three numbers were going to be the same without actually computing them.

(8) (1.4.8ab) Sketch the following curves in $\mathbb{R}^2$ and find parametrization for each.

(a) [1pt] $C : 4x^2 + y^2 = 1$,

(b) [1pt] $D : 3x + 4y = 1$.

(9) (1.5.1) Let $v = (1, 2, -3)$ and $p = (0, -2, 1)$. Evaluate the following 1-forms on the tangent vector v_p:

(a) [1pt] $y^2 dx$,

(b) [1pt] $zdy - ydz$,

(c) [1pt] $(z^2 - 1)dx - dy + x^2 dz$,

(10) [1pt] (1.5.2) If $\varphi = \sum f_i dx_i$ and $V = \sum v_i U_i$, show the the 1-form φ evaluated on the vector field V is the function $\varphi(V) = \sum f_i v_i$.

(11) (Part of 1.5.3) Evaluate the 1-form $\varphi = x^2 dx - y^2 dz$ on the vector fields

(a) [1pt] $V = xU_1 + yU_2 + zU_3$,

(b) [1pt] $xy(U_1 - U_3) + yz(U_1 - U_2)$.

(12) Express the following differentials in terms of df:

(a) [1pt] $d(f^5)$,

(b) [1pt] $d\left(\sqrt{f}\right)$, where $f > 0$,

(c) [1pt] $d(\log(1 + f^2))$.